Empirical nonextensive laws for the county distribution of total personal income and gross domestic product

Ernesto P. Borges [a,b]

[a] *Escola Politécnica, Universidade Federal da Bahia, Rua Aristides Novis, 2, 40210-630 Salvador-BA, Brazil*

[b] *Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*

Abstract

We analyze the cumulative distribution of total personal income of USA counties, and gross domestic product of Brazilian, German and United Kingdom counties, and also of world countries. We verify that generalized exponential distributions, related to nonextensive statistical mechanics, describe almost the whole spectrum of the distributions (within acceptable errors), ranging from the low region to the middle region, and, in some cases, up to the power-law tail. The analysis over about 30 years (for USA and Brazil) shows a regular pattern of the parameters appearing in the present phenomenological approach, suggesting a possible connection between the underlying dynamics of (at least some aspects of) the economy of a country (or of the whole world) and nonextensive statistical mechanics. We also introduce two additional examples related to geographical distributions: land areas of counties and land prices, and the same kind of equations adjust the data in the whole range of the spectrum.

Key words: Econophysics, Nonextensive Statistical Mechanics, Complex systems
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1 Introduction

Despite of the ubiquity of Gaussians in nature, there are many, also ubiquitous, examples of non-Gaussian distributions. Power-laws, for instance, appear

Email address: `ernesto@ufba.br` (Ernesto P. Borges).

in a variety of physical, biological, psychological, and social/economical phenomena [1,2,3,4,5,6,7,8,9]. Frequently, such systems do not exhibit power-law behavior in the entire spectrum, but rather power-law *tails*. Characterization of economical systems, more specifically the distribution of personal income, are usually assumed to follow Pareto's law [1], $p(x) \propto x^{-1-\alpha}$, in the large income region (typically $1 \geq \alpha \geq 2$), and a log-normal distribution [10],

$$p(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\log^2(x/x_0)}{2\sigma^2}\right], \tag{1}$$

in the middle (or low-middle) income region ($p(x)$ is the probability density function, x_0 is a mean value and σ^2 is a variance). See [11,12,13] for recent revisiting of this approach to the problem.

It is already known the existence of connections between economical systems (financial markets) and nonextensive statistical mechanics [14] (see [15] for a recent review). In the present work we address a different feature of economical systems: the distribution of total personal income (PI) of counties, as well as total gross domestic product (GDP) of counties for a given country (both PI and GDP can be an index for the value added). We similarly consider distribution of GDP of the *countries* of the world. We use distributions that belong to the family of the q-exponential function [16,17],

$$\exp_q(x) \equiv [1 + (1-q)x]_+^{\frac{1}{1-q}} \tag{2}$$

($q \in \mathbb{R}$, $[\cdots]_+ \equiv \max\{\cdots, 0\}$), that naturally emerge from nonextensive statistical mechanics [18,19,20] (for recent reviews and updated bibliography, see Ref. [21,22,23]). q-Exponentials (with negative argument, which is the case we are interested in here; hereafter we will consider $\exp_q(-x)$, with $q \geq 1$ and $x > 0$) present asymptotic power-law tails, as many complex systems do. Along these lines, we are able to describe (almost) the whole spectrum of the distribution (and not only the tails) with a single function, which points towards an unified approach of the problem. In a certain sense, this problem resembles another one, namely the number of citations of scientific papers, which likewise present power-law behavior only at the tail. It was first conjectured that different phenomena rule large-cited and low-cited papers (see Ref. [24] and references therein). A nonextensive approach to the problem [25] showed that it is possible to have a single function describing the whole spectrum of citations.

2 Distribution generators

Let us formulate the problem by two alternative paths. One way of characterizing a distribution is through the variational approach, in which an *entropy* is maximized under constraints of normalizability and finiteness of a generalized moment of the distribution ($\langle |x|^\gamma \rangle$ = constant, $\gamma > 0$) (see, e.g., Ref. [26]). For instance, if we take into account Boltzmann-Gibbs entropy,

$$S = -k_B \int p(x) \ln p(x)\, dx, \tag{3}$$

submitted to the constraints of normalizability

$$\int p(x)\, dx = 1 \tag{4}$$

and finiteness of a certain momentum of order γ

$$\int |x|^\gamma\, p(x)\, dx < \infty, \tag{5}$$

it comes out exponential forms (stretched exponentials),

$$p(x) \propto \exp(-\beta x^\gamma), \tag{6}$$

with β being the Lagrange multiplier. Typically $\gamma = 1$ when we are dealing with distributions of, e.g., energy, and $\gamma = 2$ for distributions of space positions, i.e., diffusion. For the sake of generality, we put not only integer values of γ, but rather $\gamma \in \mathbb{R}$; that's why it appears the modulus in Eq. (5) (although in the forthcoming examples we only consider $\gamma = 1$ or $\gamma = 2$, and also $x > 0$, which makes unnecessary the modulus). $\gamma = 1$ yields exponentials, $\gamma = 2$, Gaussians.

If, instead, we take nonextensive entropy [18]

$$S_q \equiv k \frac{1 - \int [p(x)]^q\, dx}{q-1} \qquad (q \in \mathbb{R}), \tag{7}$$

(k is a non-negative constant, related and possibly equal to Boltzmann's constant k_B), with the same normalizability constraint, Eq. (4), and a q-generalized version of the finiteness of the momentum of order γ,

$$\int |x|^\gamma\, [p(x)]^q\, dx < \infty, \tag{8}$$

then q-stretched exponentials appear:

$$\begin{aligned} p(x) &\propto \exp_q(-\beta_q\, x^\gamma) \\ &\propto [1-(1-q)\beta_q\, x^\gamma]_+^{\frac{1}{1-q}}. \end{aligned} \tag{9}$$

$q = 1$ recovers usual stretched exponentials, with $\beta_q \equiv \beta$, Eq. (6). $q \neq 1$ and $\gamma = 1$ recovers the q-exponential itself, Eq. (2). Functions with $\gamma = 2$ and $q \neq 1$ can consistently be called q-Gaussians. Particular cases are, of course, the Gaussian distribution ($q = 1$), and the Lorentzian distribution ($q = 2$). This path was followed by Ref. [27,28].

An alternative way of characterizing a distribution is through the *differential equation* it satisfies. This path was developed in Ref. [29], within the framework of nonextensive statistical mechanics, but it was originally formulated in Planck's celebrated first papers on black-body radiation law [30], the birth of quantum mechanics. See Ref. [31] for a comprehensive review on this differential equation approach, and also for historical remarks about Planck's first papers on black-body radiation. Stretched exponential distributions obey

$$\frac{1}{\gamma x^{\gamma-1}}\frac{dp}{dx} = -\beta\, p, \tag{10}$$

while nonextensive q-stretched exponentials follow a simple generalization of the former equation:

$$\frac{1}{\gamma x^{\gamma-1}}\frac{dp}{dx} = -\beta_q\, p^q, \tag{11}$$

whose solution is given by Eq. (9).

Some complex systems, e.g. re-association of oxygen in folded myoglobin [29], linguistics [32], cosmic rays [33,34,35], economical systems (distribution of returns of New York Stock Exchange [36]), and also the economical examples we are dealing here, exhibit not one, but *two* power-law regimes (i.e., two different slopes in a log-log plot, according to the value of the independent variable), with an usually marked crossover between them, sometimes referred to as the *knee*. Such behavior demands a probability distribution obeying a more general differential equation than the two former ones, namely

$$\frac{1}{\gamma x^{\gamma-1}}\frac{dp}{dx} = -(\beta_q - \beta_{q'})\, p^q - \beta_{q'}\, p^{q'} \tag{12}$$

($1 \leq q' \leq q$, $0 \leq \beta_{q'} \leq \beta_q$), where q and q' are connected to the two different slopes of the regimes (the asymptotic slope in a log-log plot is given by $\gamma/(1-q)$). q controls the slope of the first (intermediate) power-law regime, and q', the second (the tail). The solution of Eq. (12) is expressed in terms of hypergeometrical functions (see Ref. [29] for the analytical expression), and we can consistently call such functions (q, q')-stretched exponentials. They naturally present a crossover (knee) at

$$x_{knee}^{\gamma} = \frac{[(q-1)\beta_q]^{\frac{q'-1}{q-q'}}}{[(q'-1)\beta_{q'}]^{\frac{q-1}{q-q'}}}. \tag{13}$$

Notice that $\beta_{q'} = 0$, or $\beta_{q'} = \beta_q$, or even $q' = q$ recover the differential equation obeyed by q-stretched exponentials, Eq. (11), with $x_{knee} \to \infty$.

Another particular case of Eq. (12) is obtained with $q' = 1$. It is curious to note that this differential equation (with $q' = 1$) is known as Bernoulli's equation [37,38], and it is exactly the one used by Planck in his October 1900 paper [30] (with $q = 2$, $q' = 1$, $\gamma = 1$) [1]. The distribution of this $q' = 1$ case is given by [29]

$$p(x) = \left[1 - \frac{\beta_q}{\beta_1} + \frac{\beta_q}{\beta_1} e^{(q-1)\beta_1 x^{\gamma}}\right]^{\frac{1}{1-q}}, \tag{14}$$

and presents an intermediate power-law regime, followed by an *exponential* tail, with a crossover at

$$x_{knee}^{\gamma} = \frac{1}{[(q-1)\beta_1]}. \tag{15}$$

An example of system that may be described by this ($q \neq 1$, $q' = 1$) case is the measure of success of musicians [39].

[1] It is also worth mention that Planck adopted this equation as a fitting procedure (and certainly with a great amount of physical intuition). In his words [30], "one gets a radiation formula with two constants ... which, as far as I can see at the moment, fits the observational data, published up to now, as satisfactorily as the best equations put forward for the spectrum ..."

3 Geographical distribution of Total Personal Income and Gross Domestic Product

We consider Eq. (12) with $p \equiv P$ being the *inverse cumulative probability distribution*, $P(X \geq x) = \int_x^\infty dy\, p(y)$ ($P(X \geq x)$ is the probability of finding the distribution variable with a value X equal to, or greater than, x), and $x \equiv x/x_0$ is the ratio between an economical variable and its minimum value: in the discrete case, $x_i \equiv x_i/x_{min}$, where x stands for the economical variable, in our analysis, PI of a county, or GDP of a county (or of a country). Index i refers to the county (or country), and *min* is the poorest (lowest ranking) county (country).

We analyze one case of PI county distribution, USA counties (for years ranging from 1970 to 2000) [40], and three cases of GDP county distribution: Brazilian counties (from 1970 to 1996) [41], German counties (from 1992 to 1998) [42], and United Kingdom counties (from 1993 to 1998) [43]. (See [40] for the method of calculation of USA county PI.) All these cases are well described with $\gamma = 2$, i.e., (q, q')-Gaussians.

Fig. 1 illustrates the results with inverse cumulative distributions. Inverse cumulative distribution, or the rank, is equal to the number of counties $N_{counties}$ times P, with P given by the corresponding cumulative distribution probability. Three curves are shown in each Fig. 1(a)–(d): (i) q-Gaussian distributions, which can describe low range data, (ii) (q, q')-Gaussian, which shows to be able to reproduce the low-middle range, including the knee, and (iii) log-normal distributions, that were adjusted to fit middle range values. For USA and Brazil, we observe that the (q, q')-Gaussian describes the data in almost the entire range; for Germany and UK, both (q, q')-Gaussian and log-normal are able to describe the data in the low-middle region (the curves are practically visually indistinguishable in this region). For USA and Brazil, the log-normal distribution fails in the low region — see Inset of Fig.s 1(a) and 1(b). Values of the parameters are given in Table 1.

At a first glance, it might seem that log-normal distributions are more parsimonious (and consequently, preferable) in the description of these problems than the (q, q')-Gaussians, once the former has two fitting parameters, while the later has four parameters. But when we look in detail to the problem, we realize that in many cases the log-normal is able to describe just the middle range values of the distributions (sometimes low-middle range). Deciding where this middle range begins and where it ends works as if there were two additional hidden parameters in this log-normal distribution. When this happens, both log-normal and (q, q')-Gaussians present the same fitting degree of freedom.

Table 1
Parameters for the distribution functions, for the years shown in Fig. 1.

Country	Year	$N_{counties}$	q	q'	$1/\sqrt{\beta_q}$	$1/\sqrt{\beta_{q'}}$	x_0	σ
USA	2000	3110	3.80	1.7	87.71	2236.07	110	7
Brazil	1996	4973	3.50	2.1	40.82	816.50	22	10
Germany	1998	440	2.70	1.5	3.16	6.59	3.5	1.5
UK	1998	133	3.12	1.4	18.26	37.80	20	1.5

Large GDP range displays a different behavior: the distribution presents a *second* crossover, bending upwards and giving rise to a different (third) power-law regime. This effect is very pronounced for Germany, and in a smaller degree for UK, while for USA and Brazil, it is almost hidden in the binned distribution (as shown in Fig. 1), but it is visible with unbinned plots [2] . In USA, for instance, only the two major counties (Los Angeles and Cook (part of Chicago)) belong to this regime. Similarly, in Brazil, we have São Paulo and Rio de Janeiro within this regime. This feature is commonly exhibited by various systems, sometimes referred to as *king* effect [44]. It is also present in highly energetic cosmic rays, been referred to as *ankle* [33] (we adopt this nomenclature in the Figures). Such behavior is possibly related to nonequilibrium phenomena, or (at least in some cases) poor statistics, and lies outside the present approach. We recall that the number of counties in USA and Brazil is about one order of magnitude greater than that of Germany and UK, and this possibly is related to the more pronounced king effect in these last to countries.

Fig. 2 shows temporal evolution of the parameter q. USA present an approximately uniform increase of q over 30 years. In the case of Brazil, the tendency of increase from 1970 to 1990 was broken from 1990 to 1996. Germany and UK present constant values of q over the years for which there are available data. The increase of q (observed for USA and Brazil) indicates increasing *inequality*: the greater the q, long-lasting the tail, the greater the probability of finding counties much richer than others. The parameter q' (for a given country) is taken constant for all years. The smaller values of q and q' for Germany and UK, when compared to USA and Brazil, reflect the well balanced distribution of value added of these European countries, relative to the analyzed American countries. The relation between the slopes (related to q) and equality/inequality is not a new conclusion; it is known since Pareto [1] (see also Ref. [45] and references therein).

[2] In a binned distribution, the ordinate shows the number of data (normalized or not) that falls within a (usually small) region, or bin, in the abscissa. In the distributions shown in Fig. 1, the bins are logarithmic equally spaced, i.e., their width are exponentially increasing. In unbinned distributions, each point in the figure corresponds to an original data. Binned distribution was chosen in Fig. 1 for better visualization.

We have also analyzed the world GDP *country* distribution, for the year 2000 [46]. In this case, we found that a (q, q')-exponential ($\gamma = 1$) fits better the data than a (q, q')-Gaussian ($\gamma = 2$) in the low-middle region. Although the difference between the two functions (with $\gamma = 1$ and $\gamma = 2$) is perceptible, it is not that much big. This purely phenomenological observation deserves further investigation to corroborate or not our results. If it is confirmed to be $\gamma = 1$, a possible interpretation might be due to the nature of interactions between countries, which is expected to differ from the interactions between counties inside a country. Fig. 3 shows the results. The king effect is also present here, particularly for the two major GDP countries, USA and Japan.

4 Distribution of land areas and land markets

In this section we add two different examples, related to geographical distributions: (i) distribution of land areas of Brazilian counties, and (ii) distribution of Japan land prices.

Let us focus on the first example, illustrated by Fig. 4. The minor Brazilian county has 2.9 km^2 (Santa Cruz de Minas, in the State of Minas Gerais), and the major one has 161 446 km^2 (Altamira, in the State of Pará, within the Amazon forest) [47]. There are many causes for a county to have a given area, including, among others, geographical, political, demographical and economical factors. The (q, q')-Gaussian fits (within an acceptable error) practically all county areas (more than 5500, in the year 1998), from the smaller up to the greater.

Now let us consider the problem of Japan land prices, recently addressed [48]. The author found a power-law tail for the cumulative probability distribution of land price, with a slope of -1.7 ($P(X \geq x) \propto x^{-1.7}$). Fig. 5 makes evident that the q-Gaussian (with $\beta_{q'} = 0$) fits the whole range of data (except the point with the higher price) and not only the tail (we recall that the probability distribution, from [48], is binned — maybe with unbinned data, we could find $\beta_{q'} \neq 0$, and with this, it would possibly include the last point(s) under the curve).

5 Final remarks

Finally we would like to point out that q-Gaussians and log-normal distributions seem to be equally able to describe the data in the low-middle region. As these data usually range not so many decades, one cannot unequivocally

decide the 'correct' distribution by simple comparison, and such phenomenological approaches (ours included) only give hints about the underlying dynamics of the economy. q-Gaussians present power-law tails (log-normal distributions don't), which is an important characteristic when dealing with complex systems. The good concordance of the present procedure with data, including the smoothness of temporal evolution of the parameters, together with previous works along these lines [14,15], may suggest a new path for investigating economical relations, namely the development of models based on the framework of nonextensive statistical mechanics. Such q-distributions (or (q, q')-distributions) appear when long-range interactions, long-term memory, (multi)fractality and/or small-world networking are present, expected features in complex systems (including economical ones). This further central step (development of models that essentially describe the underlying microscopic dynamics of the systems) is certainly not an easy task. The answer (as suggested in Ref. [31]) may come from Barabási-Albert's approach to the problem of small-world networks [49,50], considering preferential attachment of new vertices, that are added to a network, to sites that are already well connected, leading to scale-free power-law distributions.

These examples join many others (some cited in the Introduction) that are fairly fitted by equations related to nonextensive statistical mechanics. Of course these are only empirical observations, but there is a significant amount of evidence that nonextensivity and complex systems are intimately connected (we don't mean that *all* complex systems might be somehow related to nonextensivity, but it is possible that such complex systems may be divided into classes of universality, some of them presenting a nonextensive nature). Besides, it is already known how to estimate *a priori* the q index for some classes of systems, namely low-dimensional dissipative maps, based on its dynamics (see [51,52] and references therein). Of course these are much more simple systems than the ones we are dealing with, but many efforts have being made to extend this predictive nature to other more complex systems (see also [31]). These results point towards the interpretation that the nonextensive indexes q and q' are not merely fitting parameters, but they are related to more fundamental dynamic features of the problem. With this point of view in mind, the present proposed approach, at the moment being just a phenomenological observation, may happen to be of a more fundamental nature, and not simply a fitting procedure.

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Figures

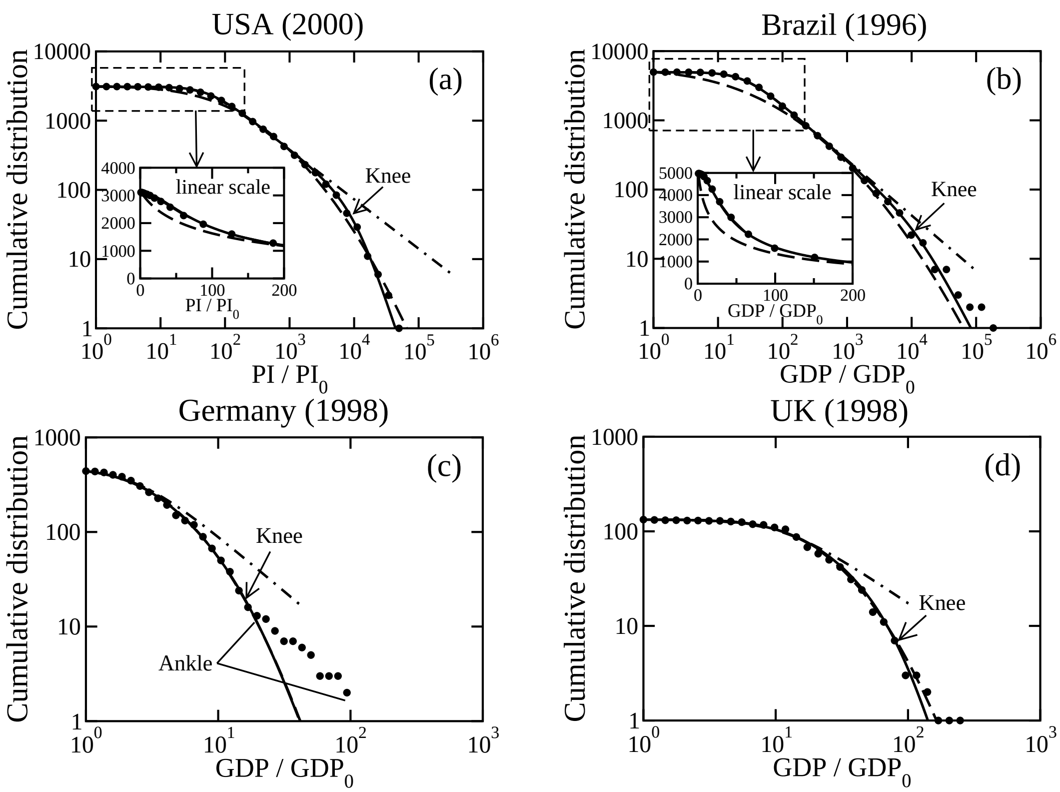

Fig. 1. Binned inverse cumulative distribution of county PI/PI_0 (USA) and GDP/GDP_0 (Brazil, Germany and UK). Three distributions are displayed for comparison: (i) q-Gaussian (with $\beta_{q'} = 0$) (dot-dashed), (ii) (q, q')-Gaussian (solid), and (iii) log-normal (dashed lines). Figures (a) and (b) present Insets with linear-linear scale, to make more evident the quality of the fitting at the low region (In Fig.s (c) and (d), the (q,q')-Gaussian and the log-normal curves are superposed and so are visually indistinguishable). The positions of the knees (according to Eq. (13)) are indicated. The ankle is particularly pronounced in (c), though it is also present in the other cases.

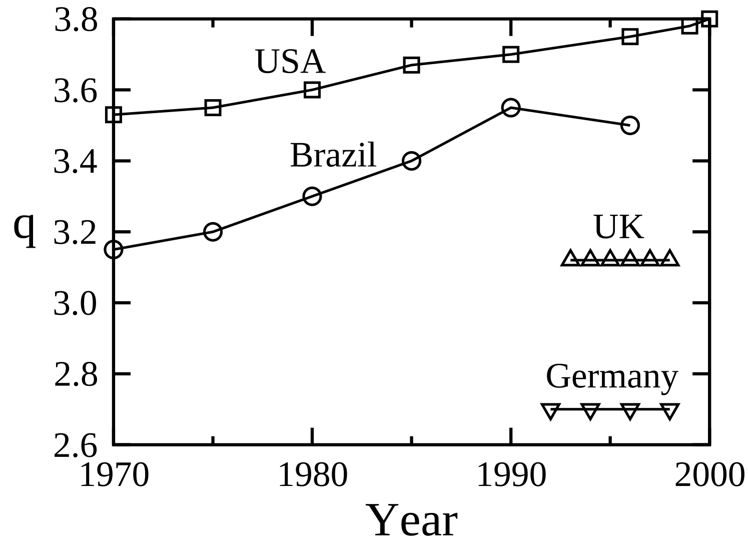

Fig. 2. Evolution of parameter q for USA (squares), Brazil (circles), UK (up triangles) and Germany (down triangles). The parameters q' (for each country) are constant for all years: $q'_{Brazil} = 2.1$, $q'_{USA} = 1.7$, $q'_{Germany} = 1.5$, $q'_{UK} = 1.4$. Lines are only guide to the eyes.

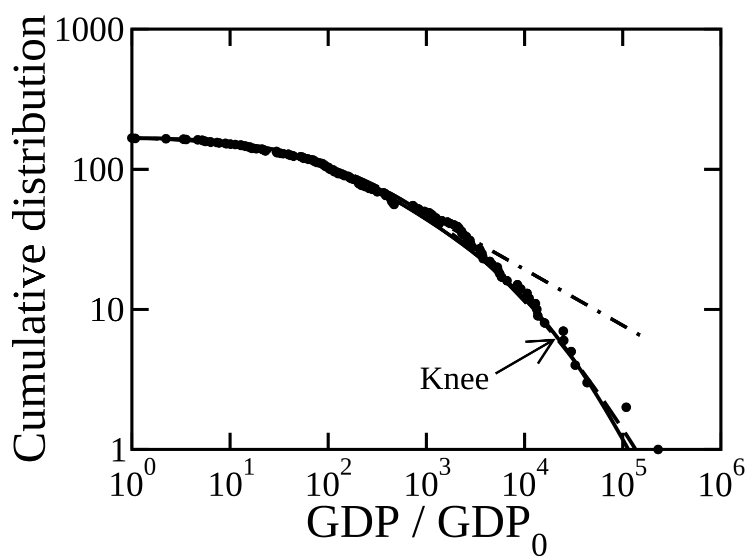

Fig. 3. Inverse cumulative distribution of GDP/GDP$_0$ of 167 *countries* for the year 2000 (unbinned data: each point corresponds to a country). The data are fitted with (q, q')-exponential (solid) and log-normal (dashed line) distributions — they are visually indistinguishable for this example. q-exponential (with $\beta_{q'} = 0$, dot-dashed) is also shown for comparison. Values of the parameters are $q = 3.5$, $q' = 1.7$, $1/\beta_q = 111.1$, $1/\beta_{q'} = 2500.0$. The knee, according to Eq. (13), is located at GDP/GDP$_0 = 19\,665$. Log-normal curve with $x_0 = 220$ and $\sigma = 13$.

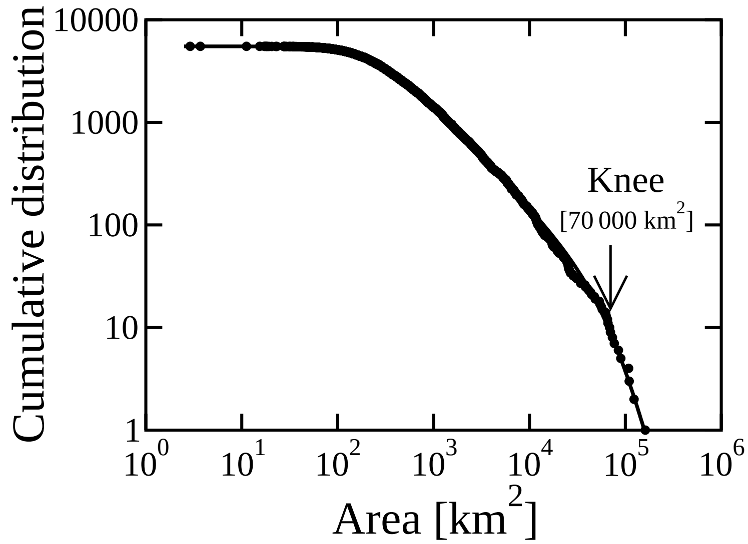

Fig. 4. Inverse cumulative distribution of land areas of Brazilian counties (unbinned data). Solid line is a (q, q')-Gaussian. $q = 3.07$, $q' = 1.56$, $1/\sqrt{\beta_q} = 353.55$ km^2, $1/\sqrt{\beta_{q'}} = 11\,226.7$ km^2.

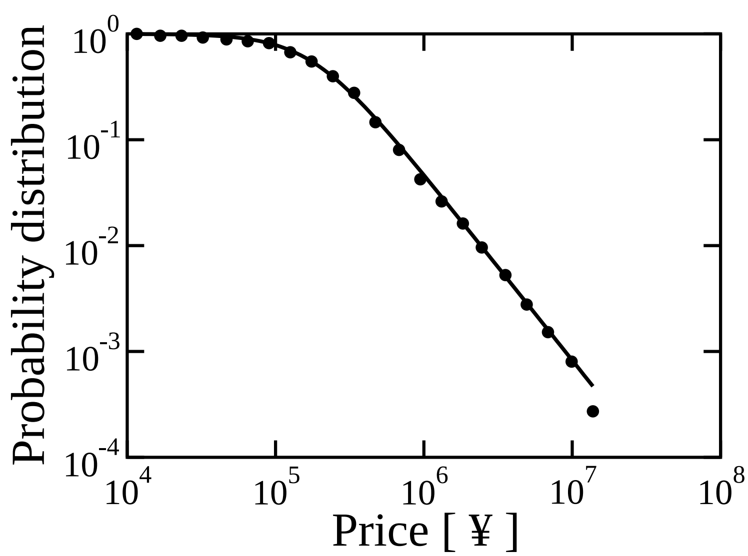

Fig. 5. Inverse cumulative probability distribution of Japan land prices for the year 1998. The data (binned) were taken from Fig. 1 of [48]. Solid curve is a q-Gaussian with $q = 2.136$, which corresponds to the slope -1.76 (found by the Author of [48]), and $1/\sqrt{\beta_q} = 188\,982$ Yen.